



**Zero discounting and optimal paths of depletion of an
exhaustible resource with an amenity value**

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Abstract

This paper studies the undiscounted utilitarian optimal paths of the canonical Dasgupta–Heal–Solow model when the stock of natural capital is a direct argument of well-being, besides consumption. We use a Keynes–Ramsey rule which yields a generalization of Hartwick's rule: if society has a zero discount rate but is ready to accept intertemporal substitution, net investment should not be zero as in the maximin case but should be positive, its level depending on the distance between the current and the long run bliss level of utility. We characterize solutions in the Cobb-Douglas utility and production case, and analyse the influence of the intertemporal elasticity of substitution on the time profile of the optimal paths. We show that, in the Cobb-Douglas case, the ratio of the values of the resource and capital stocks remains constant along the optimal path, and is independent of initial conditions.

JEL: D9, Q01, Q3

Keywords: exhaustible resources, Hartwick's rule, intertemporal substitution

Taux d'escompte nul et sentiers optimaux d'extraction d'une ressource épuisable ayant une valeur d'aménité

Nous étudions les sentiers optimaux du modèle canonique de Dasgupta–Heal–Solow, dans le cas où le taux d'escompte est nul et où le niveau du stock de capital naturel intervient dans la fonction d'utilité. Nous utilisons une règle de Keynes-Ramsey qui permet de généraliser la règle de Hartwick : si la société a un taux d'escompte nul mais accepte une substitution intertemporelle entre niveaux d'utilité, l'investissement

net ne doit plus être zéro, comme dans le cas maximin, mais doit être positif et être fixé à un niveau qui dépend de la distance entre le niveau d'utilité de court terme et le niveau de félicité atteint à long terme. Nous caractérisons les solutions optimales dans le cas où l'utilité et la production sont Cobb-Douglas et nous analysons l'influence de l'élasticité intertemporelle de substitution sur le profil temporel du sentier optimal. Nous montrons aussi que, dans ce cas Cobb-Douglas, la valeur relative des stocks de ressource et de capital reste constante sur le sentier optimal, à un niveau qui ne dépend pas des conditions initiales.

JEL : D9, Q01, Q3

Mots-clés : ressources épuisables, règle de Hartwick, substitution intertemporelle

1 Introduction

This paper studies the undiscounted utilitarian optimal paths of the canonical Dasgupta–Heal–Solow model (Dasgupta and Heal [1974], Solow [1974]) of depletion of an exhaustible resource, when the resource stock has an amenity value (Krautkraemer [1985]).

Using a zero social rate of time preference, in the spirit of Ramsey [1928], is often considered as the right ethical attitude ensuring an equal treatment of every generations, whatever their position in time. The old debate on discounting is particularly vivid when environmental issues are concerned, because they involve very long term costs and benefits, reduced to almost nothing by the usual practice of positive discounting. Very recently, the debate about the Stern Review (Stern [2007]) has largely been focused on the discounting issue, the authors being criticized for their choice of a very low utility discount rate (see for instance the debate in this review, in particular Guesnerie [2007] and Gollier [2007]).

We do find this approach appealing, however, and focus in this paper on the benchmark case of a zero discount rate.

The Dasgupta–Heal–Solow model with a zero utility discount rate has already been extensively studied in the literature, in the case where the sole source of well-being is consumption. Dasgupta and Heal [1979] provide a complete solution when the production function is Cobb-Douglas and the utility function CRRA, with an intertemporal elasticity of substitution lower than unity to ensure that utility is bounded above. Mitra [1980] studies the same problem in a very general framework, without specifying the production and utility functions. Under usual regularity requirements and the assumption that utility is bounded above, he expresses the asymptotic properties of

the model: consumption and the price of the exhaustible resource in terms of the consumption good increase monotonically to infinity. He then shows that the production and utility functions must satisfy jointly conditions ensuring that “as the resource stock is (rapidly) depleted, consumption can still grow fast enough to make the utility sum converge”.

We maintain that considering natural resources – even exhaustible ones – only as inputs in the production process is inadequate, and that acknowledging their amenity value is essential. The idea may not seem so obvious for resources such as fossil fuels, but their extraction and use are polluting and contribute to global warming, which justifies considering that the stock not yet extracted has an amenity value: see d’Autume, Hartwick and Schubert [2008]. Much more generally, numerous items of natural capital are unavoidably destroyed by economic development, whereas they clearly have an amenity value. Biodiversity appears as a perfect exemple of this phenomenon. This leads us to consider that the exhaustible resource of the Dasgupta–Heal–Solow model can be more broadly interpreted as natural capital, having an amenity value, and the depletion of which is largely irreversible.

This model has been studied by Krautkraemer [1985] with a discounted utilitarian social welfare function, while d’Autume and Schubert [2008], in a companion paper, study the maximin paths. We here extend this analysis to the more general case of a zero discount rate.

We establish a Keynes–Ramsey–Hartwick rule, which appears as a generalization of Hartwick’s rule. It states that if society has a zero discount rate but is ready to accept intertemporal substitution, net investment should not be zero as in the maximin case but should be positive, its level depending on the distance between the current and the long run value of utility. The maximin path appears as a limit case of the undiscounted utilitarian one, when society does not accept any intertemporal substitution.

We then specify the form of the production and utility functions. The production function is Cobb–Douglas, as in Dasgupta and Heal [1979]. On the utility side, we want to disentangle the effects of intertemporal and intratemporal substitutability between consumption and amenity. To this effect, we define a composite consumption index, combining consumption and the natural capital stock; instantaneous utility is a CRRA function of this index, with an intertemporal elasticity of substitution lower than unity. We focus on the case where the composite index is of the Cobb–Douglas form, that is when intratemporal substitutability of consumption and amenity in utility is high enough. We show that the exhaustible resource is totally depleted in the long run, and provide an explicit solution of the model, thus showing that the capital path is quasi-arithmetic. A new striking result is that, under these assumptions, the ratio of the values of the resource and capital stocks

is constant along the optimal path and is independent of initial conditions.

2 The model

2.1 The general case

We consider the standard Dasgupta–Heal–Solow model with an exhaustible resource but assume, following Krautkraemer [1985], that the stock of resource has an amenity value for consumers and therefore appears in the utility function. The exhaustible resource may be interpreted, in a broad sense, as natural capital.

Instantaneous utility $U(c, X)$ depends on physical consumption c and the stock of resource X . Production $Y = F(K, x)$ is a function of the capital stock K and the flow x of exhaustible resource extracted at a given time. The production function is increasing in its two arguments. It has decreasing returns to scale, which follows from the implicit presence of a given and constant stock of labor.

We consider the following problem:

$$\max \int_0^\infty [U(c_t, X_t) - U^*] dt \quad (1)$$

$$\dot{K}_t = F(K_t, x_t) - c_t, \quad (2)$$

$$\dot{X}_t = -x_t, \quad (3)$$

$$K_0 \text{ and } X_0 \text{ given.} \quad (4)$$

We assume zero discounting and follow the approach introduced by Ramsey [1928]. U^* is the constant long run level of utility. The integrand thus tends to zero, which makes it possible for the integral to remain finite. We shall focus on a case where U^* is equal to zero.

To this end we specify the utility function as

$$U(c, X) = \frac{u(c, X)^{1-1/\sigma}}{1-1/\sigma}, \quad 0 < \sigma < 1, \quad (5)$$

where $u(c, X)$ is the composite consumption index, characterizing intratemporal preferences for consumption and resource amenity, and σ is the intertemporal elasticity of substitution. Function $u(c, X)$ is increasing in both arguments and has usual properties, namely is quasi-concave. We assume elasticity σ to be lower than unity. Then $U(c, X)$ is negative and tends to

zero if $u(c, X)$ tends to plus infinity. We then assume ex ante that there exist feasible paths such that $u(c, X)$ tends to infinity as t tends to infinity. We derive optimality conditions, characterize the potential optimal solution and check ex post that this is indeed the case for our specification of the production and utility functions.

We assume a Cobb-Douglas production function:

$$Y = F(K, x) = K^\alpha x^\beta, \quad \alpha, \beta > 0, \quad \alpha + \beta < 1. \quad (6)$$

Production cannot be positive without resource use x being strictly positive: thus the resource is *necessary* for production, in the terminology of Dasgupta and Heal [1979]. As it is exhaustible and faces unavoidable depletion, no permanent level of consumption nor utility is sustainable without physical capital growing without limit in order to compensate for decreasing reliance on the resource. The long run allocation will be characterized by capital K tending to infinity and resource use x tending to zero.

2.1.1 The Keynes-Ramsey-Hartwick rule

Let

$$q = F_x(K, x) \quad (7)$$

be the (shadow) price of the resource.

We define net investment, or genuine saving, and Net National Product as

$$I_{net} = \dot{K} + q\dot{X} = \dot{K} - qx, \quad (8)$$

$$Y_{net} = Y + q\dot{X} = Y - q\dot{X}. \quad (9)$$

Both take into account the negative effect of the depletion of the natural resource stock. Net National Product is the sum of consumption and Net Investment: $Y_{net} = c + I_{net}$. Note that with Cobb-Douglas production, Net National Product is a constant share of national product as $q = \beta Y/x$ and therefore $Y_{net} = (1 - \beta)Y$.

Proposition 1 (i) *The optimal path satisfies the Keynes-Ramsey-Hartwick rule*

$$I_{net} = \dot{K} + q\dot{X} = \frac{U^* - U(c, X)}{U_c(c, X)}, \quad (10)$$

with $U^* = 0$.

(ii) The allocation of Net National Product on the optimal path is described by the following rule:

$$I_{net} = \frac{\sigma}{\sigma + (1 - \sigma)\eta_c(c, X)} Y_{net}, \quad (11)$$

$$c = \frac{(1 - \sigma)\eta_c(c, X)}{\sigma + (1 - \sigma)\eta_c(c, X)} Y_{net}. \quad (12)$$

where $\eta_c(c, X)$ is the elasticity of $u(c, X)$ with respect to consumption.

Proof. Let $V(K, X)$ be the value function of problem (1)–(4). Bellman equation is (dropping the time index):

$$0 = \max_{c, x} [U(c, X) - U^*] + V_K(K, X) [F(K, x) - c] - V_X(K, X)x. \quad (13)$$

As explained above U^* will eventually be zero but interpretation is clearer if we make it appear.

First order optimality conditions are:

$$U_c(c, X) = V_K(K, X), \quad (14)$$

$$V_K(K, X)F_x(K, x) = V_X(K, X). \quad (15)$$

If we put them back in the Bellman equation, we obtain

$$0 = U(c, X) - U^* + U_c(c, X) [F(K, x) - c - xF_x(K, x)]. \quad (16)$$

As $F(K, x) - c - xF_x(K, x) = \dot{K} - qx$, this proves part (i) of the proposition.

As $U = u^{1-1/\sigma}$, we have

$$\frac{U_c(c, X)}{U(c, X)} = \left(1 - \frac{1}{\sigma}\right) \frac{u_c(c, X)}{u(c, X)} = \left(1 - \frac{1}{\sigma}\right) \frac{\eta_c(c, X)}{c}$$

where $\eta_c(c, X)$ is the elasticity of utility with respect to consumption. Taking into account $U^* = 0$, equation (10) becomes

$$\dot{K} + q\dot{X} = \frac{\sigma}{1 - \sigma} \frac{c}{\eta_c(c, X)}.$$

(11) and (12) directly follow. This completes the proof. ■

Hartwick's rule (Hartwick [1977]) states that net investment should be zero so that society should invest in physical capital accumulation the rents

obtained from the extraction of the exhaustible resource. Equivalently, society should consume its Net National Product. As shown by Solow [1974], this is indeed the optimal rule if society adopts a Rawlsian – or maximin – social objective function. On a regular path, this will ensure that utility remains constant.

In his pioneering article on optimal economic growth, Ramsey [1928] assumes that the utility function $U(c)$ is bounded and reaches a maximum bliss level U^* for a finite or infinite level of consumption. This enables him to treat the case of a zero discount rate, which appears to him as the only ethical choice. The objective of the social planner is to maximise the sum of the undiscounted gaps $U(c) - U^*$ between current utility and the bliss level U^* . Ramsey then shows that current investment is linked to current capital by an explicit relationship:

$$\dot{K} = \frac{U^* - U(c)}{U'(c)}.$$

Ramsey acknowledges the help of Keynes to provide an intuitive interpretation of this rule which thus became known as the Keynes-Ramsey rule¹.

Equation (10) appears as a generalization of both the Hartwick and the Keynes-Ramsey rules. If society has a zero discount rate but is ready to accept intertemporal substitution, net investment should not be zero but positive. According to the Keynes-Ramsey rule, its level depends on the distance to the stationary point. More precisely, its value expressed in terms of utility $U_c(c, X) [\dot{K} + q\dot{X}]$ is equal to the distance $U^* - U(c, X)$ between current utility and its long run value. Thus, the farther the economy from the stationary point, the higher its net investment *when it is expressed in terms of utility*.

The maximin case is covered by proposition 1 and corresponds to the case $\sigma = 0$. We stressed in d'Autume and Schubert [2008] that the maximin case can be seen as the limit case of a zero intertemporal elasticity of substitution. Society then refuses any intertemporal substitution and, in regular cases, utility remains constant over time. As shown by proposition 1, we recover the strict Hartwick's rule:

$$\dot{K} + q\dot{X} = 0, \quad c = Y_{net}.$$

Equations (11) and (12) show how Net National Product should be shared between net investment and consumption. The presence of the elasticity

¹Note that, contrary to the standard usage, the Keynes-Ramsey rule is much more specific than the Euler condition of optimal consumption behavior.

$\eta_c(c, X)$ implies that the shares vary through time. The equations nevertheless suggest that a higher elasticity of substitution induces to invest more, and therefore to consume less in the short run, and more in the longer run. Society indeed accepts more intertemporal substitution.

2.1.2 The other optimality conditions and the dynamic system

We now proceed with the other optimality conditions.

The envelop theorem allows us to obtain the evolution of the shadow prices. Let

$$\lambda = U_c(c, X) = V_K(K, X), \quad \mu = V_X(K, X). \quad (17)$$

The price of the resource stock in terms of capital is

$$q = \mu/\lambda, \quad (18)$$

as can be checked from equations (15) and (7). Differentiating the Bellman equation (13) with respect to K , we get

$$0 = V_{KK}(K, X) [F(K, x) - c] + V_K(K, X) F_K(K, x) - V_{KX}(K, X)x$$

i.e.

$$\dot{\lambda} + F_K \lambda = 0, \quad (19)$$

and, differentiating with respect to X ,

$$0 = U_X(c, X) + V_{KX}(K, X) [F(K, x) - c] - V_{XX}(K, X)x$$

i.e.

$$U_X(c, X) + \dot{\mu} = 0.$$

We deduce the evolution of q ,

$$\frac{\dot{q}}{q} = F_K - \frac{1}{q} \frac{U_X(c, X)}{U_c(c, X)}. \quad (20)$$

From the definition of U , the marginal rate of substitution U_X/U_c only depends on the $u(c, X)$ function and is equal to u_X/u_c . This MRS is equal to the (shadow) price of the amenity in terms of the produced good. Equation (20) reads

$$\frac{\dot{q}}{q} + \frac{1}{q} \frac{u_X(c, X)}{u_c(c, X)} = F_K. \quad (21)$$

and appears as a modified version of the Hotelling rule, where the return of the resource now includes its relative amenity value.

In the Cobb-Douglas production case, the dynamic system is the following:

$$\dot{K} = Y - c \quad (22)$$

$$\dot{X} = -\beta \frac{Y}{q} \quad (23)$$

$$q = \beta Y^{1-\frac{1}{\beta}} K^{\frac{\alpha}{\beta}} \quad (24)$$

$$c = \frac{(1-\sigma)\eta_c(c, X)}{\sigma + (1-\sigma)\eta_c(c, X)} (1-\beta)Y \quad (25)$$

$$\frac{\dot{q}}{q} = \alpha \frac{Y}{K} - \frac{1}{q} \frac{u_X(c, X)}{u_c(c, X)}. \quad (26)$$

Unknowns are K , X , Y , c and q . The first two equations describe the physical constraints. (24) comes from the inversion of the production function together with the definition of q . (25) is derived from the Keynes-Ramsey-Hartwick rule, while (26) is the Hotelling rule (21).

The shadow price of consumption λ and the Euler equation (19) describing its evolution have disappeared. The Hartwick rule, together with the Hotelling rule, is sufficient to describe optimal consumption choices.

2.2 Cobb-Douglas composite consumption index

We now assume that the composite consumption index is also Cobb-Douglas:

$$u(c, X) = c^\omega X^{1-\omega}, \quad 0 < \omega < 1. \quad (27)$$

The elasticity of utility with respect to consumption is then $\eta_c = \omega$, and equations (12) and (11) describing how Net National Product Y_{net} is shared between consumption and investment along the optimal path now read, with $Y_{net} = (1-\beta)Y$,

$$c = (1-\hat{\beta})Y, \quad (28)$$

$$\dot{K} = \hat{\beta}Y, \quad (29)$$

with

$$1 - \hat{\beta} = (1-\beta) \frac{(1-\sigma)\omega}{\sigma + (1-\sigma)\omega}. \quad (30)$$

$\hat{\beta}$ lies between β and 1, and is equal to β when $\sigma = 0$. Moreover, $q\dot{X} = -qx = -\beta Y$, so that genuine saving is

$$I_{net} = \dot{K} + q\dot{X} = (\hat{\beta} - \beta)Y > 0. \quad (31)$$

Consumption, investment and net investment all are constant shares of the gross national product. In the maximin case, $\sigma = 0$ and $\widehat{\beta} = \beta$ so that net investment is equal to zero. When $\sigma > 0$, net investment is positive and even growing, as we shall check that product Y is increasing. This of course does not contradict the Keynes-Ramsey-Hartwick rule as marginal utility U_c decreases so that net investment expressed in terms of utility decreases to zero.

2.2.1 The asymptotic properties

Proposition 2 *With Cobb-Douglas functional forms for production $F(K, x)$ and the consumption index $u(c, X)$,*

- (i) *an optimal path exists iff $\alpha > \widehat{\beta}$;*
- (ii) *assume $\alpha > \widehat{\beta}$, then (a) the capital stock grows without bounds, and (b) the resource stock is asymptotically exhausted.*

Proof. Let $W_K = \lambda K$ be the value of the capital stock. As the saving rate is constant in the case of a Cobb-Douglas composite consumption index (equation (29)), equation (19) reads

$$\frac{\dot{\lambda}}{\lambda} = -F_K = -\alpha \frac{Y}{K} = -\frac{\alpha}{\widehat{\beta}} \frac{\dot{K}}{K}.$$

Then the shadow price of capital can be expressed as a function of the sole capital stock:

$$\lambda = B_1 K^{-\frac{\alpha}{\widehat{\beta}}}$$

where $B_1 \neq 0$ is a constant, and the value of the capital stock is

$$W_K = \lambda K = B_1 K^{1-\frac{\alpha}{\widehat{\beta}}}. \quad (32)$$

W_K tends to zero as time tends to infinity iff $\alpha < \widehat{\beta}$ and K tends to zero, or $\alpha > \widehat{\beta}$ and K tends to infinity. If the capital stock were to tend to zero, production and consumption would do the same as the resource input also has to tend to zero. This cannot be optimal in a model with zero discounting. Then an optimal path exists if and only if $\alpha > \widehat{\beta}$. This proves part (i) of the proposition. Along this optimal path, the capital stock grows without limit, in order to maintain an increasing consumption in spite of the decrease in resource use. This proves part (ii a) of the proposition.

Let now $W_X = \mu X$ be the value of the resource stock. From equations (18), (19) and (21) and with the Cobb-Douglas production and composite

consumption index, we have

$$\frac{\dot{\mu}}{\mu} = -\frac{u_X/u_c}{F_x} = -\frac{1-\omega}{\omega} \frac{c/X}{\beta Y/x} = \frac{1-\omega}{\omega} \frac{1}{\beta} \frac{c}{Y} \frac{\dot{X}}{X}.$$

As the marginal propensity to consume is constant in the case of a Cobb-Douglas composite consumption index (equation (28)),

$$\frac{\dot{\mu}}{\mu} = \frac{1-\omega}{\omega} \frac{1-\hat{\beta}}{\beta} \frac{\dot{X}}{X}.$$

Then, in symmetry to the case of the capital stock, the shadow price of the resource can be expressed as a function of the sole resource stock:

$$\mu = B_2 X^{\frac{1-\omega}{\omega} \frac{1-\hat{\beta}}{\beta}}$$

where $B_2 \neq 0$ is a constant, and the value of the resource stock is

$$W_X = \mu X = B_2 X^{\hat{\phi}} \quad (33)$$

with

$$\hat{\phi} = 1 + \frac{1-\omega}{\omega} \frac{1-\hat{\beta}}{\beta} = 1 + (1-\sigma) \frac{1-\omega}{\sigma + (1-\sigma)\omega} \frac{1-\beta}{\beta}. \quad (34)$$

W_X tends to zero as time tends to infinity if and only if X tends to zero. This proves part (ii b) of the proposition. ■

The condition

$$\alpha > \hat{\beta} \iff \alpha > \frac{\sigma + \beta(1-\sigma)\omega}{\sigma + (1-\sigma)\omega}$$

was first identified by Dasgupta and Heal [1979], for the case without resource amenity, where it reduces to $\alpha > \sigma + \beta(1-\sigma)$. It is more stringent than the condition $\alpha > \beta$ required in the maximin consumption case (Solow [1974]) or the maximin utility case (d'Autume and Schubert [2008]). Moreover, the condition involves technological parameters only in the maximin case, whereas it involves here both technological and preferences parameters. The higher the preference for amenity (the smaller ω), the more stringent the condition on α is. Intuitively, this is a joint condition on parameters ensuring that if the amenity value of the resource stock is high, the share of the resource flow in production is low, and vice versa. The economy cannot grow without bounds if the resource has a high amenity value *and* a high share in production.

2.2.2 The dynamic system

The dynamic system (22)–(26) now reads

$$\dot{K} = \widehat{\beta}Y \quad (35)$$

$$\dot{X} = -\beta \frac{Y}{q} \quad (36)$$

$$q = \beta Y^{1-\frac{1}{\beta}} K^{\frac{\alpha}{\beta}} \quad (37)$$

$$\frac{\dot{q}}{q} = \alpha \frac{Y}{K} - \frac{1}{q} \frac{1-\omega}{\omega} (1-\widehat{\beta}) \frac{Y}{X}. \quad (38)$$

Using (35) and (36), the Hotelling rule (38) can be written as:

$$\frac{\dot{q}}{q} = \frac{\alpha}{\widehat{\beta}} \frac{\dot{K}}{K} + \frac{1-\omega}{\omega} \frac{1-\widehat{\beta}}{\beta} \frac{\dot{X}}{X} = \frac{\alpha}{\widehat{\beta}} \frac{\dot{K}}{K} + (\widehat{\phi} - 1) \frac{\dot{X}}{X},$$

where $\widehat{\phi}$ is defined in equation (34). This shows that a linear combination of q , K and X remains constant, so that we have

$$q = \Phi_0 K^{\frac{\alpha}{\beta}} X^{\widehat{\phi}-1}, \quad (39)$$

with

$$\Phi_0 = q_0 K_0^{-\frac{\alpha}{\beta}} X_0^{1-\widehat{\phi}}. \quad (40)$$

2.2.3 The optimal solution

As in d'Autume and Schubert [2008] for the maximin case, the solution is obtained by time elimination and variable separation.

Proposition 3 *With Cobb-Douglas functional forms for production $F(K, x)$ and the consumption index $u(c, X)$,*

(i) *an aggregate of the capital and natural resource stocks is conserved along the optimal path:*

$$K^{\frac{\alpha-\widehat{\beta}}{\widehat{\beta}\widehat{\phi}}} X = K_0^{\frac{\alpha-\widehat{\beta}}{\widehat{\beta}\widehat{\phi}}} X_0; \quad (41)$$

(ii) *equivalently, the ratio of the values of the resource and capital stocks is conserved along the optimal path and its value is independent of initial conditions:*

$$\frac{W_X}{W_K} = \frac{qX}{K} = \frac{\beta\widehat{\phi}}{\alpha-\widehat{\beta}}; \quad (42)$$

(iii) *the optimal capital stock is*

$$K_t = \left[(1-\pi)A_0 t + K_0^{1-\pi} \right]^{\frac{1}{1-\pi}}, \quad (43)$$

where

$$A_0 = \widehat{\beta} \left(\frac{\alpha - \widehat{\beta}}{\widehat{\phi}} X_0 K_0^{\frac{\alpha - \widehat{\beta}}{\widehat{\beta}\widehat{\phi}}} \right)^{\frac{\beta}{1-\beta}}, \quad (44)$$

$$\pi = \left(\frac{\alpha - \beta}{\beta} - \frac{\alpha - \widehat{\beta}}{\widehat{\beta}\widehat{\phi}} \right) \frac{\beta}{1-\beta}, \quad 0 < \pi < \alpha; \quad (45)$$

(iv) the optimal consumption index is an increasing function of the capital stock:

$$u = u_0 (K/K_0)^\chi, \quad (46)$$

where

$$u_0 = (1 - \widehat{\beta})^\omega \left(\frac{\alpha - \widehat{\beta}}{\widehat{\phi}} \right)^{\frac{\beta\omega}{1-\beta}} K_0^{\frac{(\alpha-\beta)\omega}{1-\beta}} X_0^{\frac{\beta\omega}{1-\beta} + 1 - \omega}, \quad (47)$$

$$\chi = \omega\pi - (1 - \omega) \frac{\alpha - \widehat{\beta}}{\widehat{\beta}\widehat{\phi}} > 0; \quad (48)$$

(v) the value function is

$$V(K, X) = B \left(K^{\frac{\alpha - \widehat{\beta}}{\widehat{\beta}\widehat{\phi}}} X \right)^\psi, \quad (49)$$

where

$$B = -\frac{\sigma}{1-\sigma} \frac{(1 - \widehat{\beta})^{-\omega \frac{1-\sigma}{\sigma}}}{\alpha - \widehat{\beta}} \left(\frac{\alpha - \widehat{\beta}}{\widehat{\phi}} \right)^{-\frac{\beta}{1-\beta} \frac{\sigma + (1-\sigma)\omega}{\sigma}} < 0, \quad (50)$$

$$\psi = -\frac{\omega(1-\sigma) + \sigma}{\sigma} \beta \widehat{\phi} = \frac{\beta}{1-\beta} \frac{\omega(1-\sigma) + \sigma}{\sigma} - \frac{(1-\omega)(1-\sigma)}{\sigma} < 0 \quad (51)$$

Proof. The ratio of equations (36) and (35) yields, using (39):

$$\frac{dX}{dK} = -\frac{\beta}{\widehat{\beta}} \frac{1}{q} = -\frac{1}{\Phi_0} \frac{\beta}{\widehat{\beta}} K^{-\frac{\alpha}{\widehat{\beta}}} X^{1-\widehat{\phi}}.$$

Thus we obtain

$$X^{\widehat{\phi}-1} dX = -\frac{1}{\Phi_0} \frac{\beta}{\widehat{\beta}} K^{-\frac{\alpha}{\widehat{\beta}}} dK. \quad (52)$$

Integration of equation (52) yields

$$\frac{1}{\widehat{\phi}} \left(X_0^{\widehat{\phi}} - X^{\widehat{\phi}} \right) = \frac{1}{\Phi_0} \frac{\beta}{\alpha - \widehat{\beta}} \left(K_0^{1-\frac{\alpha}{\widehat{\beta}}} - K^{1-\frac{\alpha}{\widehat{\beta}}} \right). \quad (53)$$

Making $X \rightarrow 0$ and $K \rightarrow \infty$ in equation (53) then yields, as $\alpha > \widehat{\beta}$,

$$\frac{1}{\widehat{\phi}} X_0^{\widehat{\phi}} = \frac{1}{\Phi_0} \frac{\beta}{\alpha - \widehat{\beta}} K_0^{1-\frac{\alpha}{\widehat{\beta}}}, \quad (54)$$

which determines Φ_0 and therefore q_0 , by equation (40):

$$q_0 = \Phi_0 K_0^{\frac{\alpha}{\widehat{\beta}}} X_0^{\widehat{\phi}-1} = \frac{\beta \widehat{\phi}}{\alpha - \widehat{\beta}} \frac{K_0}{X_0}. \quad (55)$$

We also have, from (53),

$$\frac{1}{\widehat{\phi}} X^{\widehat{\phi}} = \frac{1}{\Phi_0} \frac{\beta}{\alpha - \widehat{\beta}} K^{1-\frac{\alpha}{\widehat{\beta}}}. \quad (56)$$

Dividing side by side equations (56) and (54) allows us to obtain

$$\left(\frac{X}{X_0} \right)^{\widehat{\phi}} = \left(\frac{K}{K_0} \right)^{1-\frac{\alpha}{\widehat{\beta}}}, \quad (57)$$

which yields (41). This proves part (i) of the proposition.

From (39), (41) and (55), we obtain

$$\frac{qX}{K} = \Phi_0 K^{\frac{\alpha}{\widehat{\beta}}-1} X^{\widehat{\phi}} = \Phi_0 K_0^{\frac{\alpha}{\widehat{\beta}}-1} X_0^{\widehat{\phi}} = \frac{\beta \widehat{\phi}}{\alpha - \widehat{\beta}}$$

This proves part (ii) of the proposition.

We provide in the Appendix a proof of the remaining parts of the proposition. ■

Equation (41) defines a family of trajectories in the (K, X) plane. Initial endowments (K_0, X_0) and, more precisely, their aggregate $K_0^{\frac{\alpha-\widehat{\beta}}{\widehat{\beta}\widehat{\phi}}} X_0$ determine the relevant trajectory. The economy follows this curve in a downward direction, as man-made capital substitutes for natural capital. The capital stock tends to infinity, as the resource stock tends to zero.

The capital stock follows a quasi-arithmetic path. So does the resource stock, which is linked to the capital stock by equation (41), and the extraction level, which is given by

$$x = Y^{1/\beta} K^{-\alpha/\beta} = (A_0/\widehat{\beta})^{\frac{1}{\beta}} K^{\frac{\pi-\alpha}{\beta}}.$$

As K increases with time whereas x has to be decreasing, π has to be smaller than α , which we check in the Appendix.

The composite consumption index $u(c, X)$ is an increasing function of K and therefore increases without limit along the optimal path. Utility $U = u^{1-1/\sigma}/(1 - 1/\sigma)$ increases and tends to zero as K and time tend to infinity. We have indeed checked ex post that $U^* = 0$.

In this Cobb-Douglas production and utility case, the value-function $V(K, X)$ is also Cobb-Douglas. This explains the striking property of constancy of the relative value of the stocks of capital and resource along the optimal path. q is the MRS V_X/V_K . With a Cobb-Douglas function, the ratio

$$\frac{qX}{K} = \frac{V_X X}{V_K K}$$

obviously remains constant at a level equal to the ratio of the elasticities of the value-function. This level does not depend on initial conditions.

By definition, the elasticity of substitution controls the amount of intertemporal substitution. We check in the Appendix that the higher σ the lower u_0 and the higher the growth rate of utility. Thus an economy with a high σ accepts to sacrifice current utility in order to increase future utility. To the contrary, an economy with a zero σ chooses a constant utility level. Note that this maximin behaviour amounts to favoring utilities of current generations. This runs against the idea that a model with an exhaustible resource is necessarily of a cake-eating type, and that maximin behaviour then leads to preserving the utilities of future generations. When society is able to accumulate physical capital, it is capable of unlimited growth of consumption and utility, even in the presence of an exhaustible resource. This was indeed one of the main results of Dasgupta and Heal [1979] and Mitra [1980].

As already mentioned, the maximin utility case corresponds to the case where the intertemporal elasticity of substitution is zero. Utility is constant and we recover the results of d'Autume and Schubert [2008]. Our point of departure was then to assume an arbitrary discount rate of utility ρ , but we showed that the influence of this discount rate vanished as we let σ tend to zero. If society does not accept any substitution between the welfares of different generations, then the weights it attributes to these generations are immaterial. We may as well suppose them equal and assume the discount rate to be zero.

The case without resource amenity ($\omega = 1$) is the case treated in Dasgupta and Heal [1979].

3 Conclusion

An obvious desirable extension would be to depart from the Cobb-Douglas composite consumption case. If consumption and the amenity were less substitutable in well-being, society might wish to preserve forever a positive stock of resource. This is the result we obtain in the maximin case (d’Autume and Schubert [2008]). It might be possible to determine an endogenous long run level of the resource stock in the more general zero discount setting. This raises theoretical issues for the Ramsey approach, as the bliss level would be endogenous.

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Appendix

Proof of parts (iii), (iv) and (v) of proposition 3

We characterize the evolution of K to prove part (iii) of proposition 3.

From (41) we have

$$X = X_0 \left(\frac{K}{K_0} \right)^{-\frac{\alpha - \hat{\beta}}{\hat{\beta}\hat{\phi}}}.$$

From (42) and (37) we obtain

$$\frac{\hat{\beta}\hat{\phi}}{\alpha - \hat{\beta}} \frac{K}{X} = \beta Y^{1-\frac{1}{\beta}} K^{\frac{\alpha}{\beta}}.$$

Eliminating X between these two equations yields:

$$Y = \left(\frac{\alpha - \hat{\beta}}{\hat{\phi}} K_0^{\frac{\alpha - \hat{\beta}}{\hat{\beta}\hat{\phi}}} X_0 K^{\frac{\alpha - \beta}{\beta} - \frac{\alpha - \hat{\beta}}{\hat{\beta}\hat{\phi}}} \right)^{\frac{\beta}{1-\beta}},$$

and therefore

$$\frac{Y}{Y_0} = \left(\frac{K}{K_0} \right)^{\pi} \quad (58)$$

with

$$Y_0 = \left(\frac{\alpha - \hat{\beta}}{\hat{\phi}} \right)^{\frac{\beta}{1-\beta}} K_0^{\frac{\alpha - \beta}{1-\beta}} X_0^{\frac{\beta}{1-\beta}}. \quad (59)$$

We thus obtain

$$\dot{K} = \hat{\beta}Y = \hat{\beta}Y_0 K_0^{-\pi} \stackrel{def}{=} A_0 K^{\pi},$$

with

$$\pi = \left(\frac{\alpha - \beta}{\beta} - \frac{\alpha - \hat{\beta}}{\hat{\beta}\hat{\phi}} \right) \frac{\beta}{1 - \beta}. \quad A_0 = \hat{\beta} \left(\frac{\alpha - \hat{\beta}}{\hat{\phi}} K_0^{\frac{\alpha - \hat{\beta}}{\hat{\beta}\hat{\phi}}} X_0 \right)^{\frac{\beta}{1-\beta}}.$$

π is positive as $\beta < \hat{\beta}$ and $\hat{\phi} > 1$ imply $\frac{\alpha-\beta}{\beta} > \frac{\alpha-\hat{\beta}}{\hat{\beta}\hat{\phi}}$. π is also smaller than α :

$$\pi - \alpha = -\frac{\beta}{1-\beta} \left((1-\alpha) + \frac{\alpha-\hat{\beta}}{\hat{\beta}\hat{\phi}} \right) < 0.$$

Equation (43) follows.

We now turn to the expression of u (part (iv) of proposition 3).

Using (27) and (28), we have

$$u = c^\omega X^{1-\omega} = \left((1-\hat{\beta})Y \right)^\omega X^{1-\omega},$$

and therefore

$$\frac{u}{u_0} = \left(\frac{Y}{Y_0} \right)^\omega \left(\frac{X}{X_0} \right)^{1-\omega}$$

or, using (58) and (41),

$$\frac{u}{u_0} = \left(\frac{K}{K_0} \right)^{\omega\pi} \left(\frac{K}{K_0} \right)^{-(1-\omega)\frac{\alpha-\hat{\beta}}{\hat{\beta}\hat{\phi}}} \stackrel{def}{=} \left(\frac{K}{K_0} \right)^\chi.$$

On the other hand,

$$u_0 = \left((1-\hat{\beta})Y_0 \right)^\omega X_0^{1-\omega},$$

which, using (59), yields the value of u_0 which appears in part (iv) of the proposition.

We finally compute the value function V (part (v) of proposition 3).

The value function is

$$\begin{aligned} V(K_0, X_0) &= \max \int_0^\infty [U(c_t, X_t) - U^*] dt \\ &= \max \frac{1}{1-\frac{1}{\sigma}} \int_0^\infty u_t^{1-\frac{1}{\sigma}} dt, \end{aligned}$$

i.e., using (46) and (43),

$$V(K_0, X_0) = \frac{u_0^{1-\frac{1}{\sigma}} K_0^{-\chi(1-\frac{1}{\sigma})}}{1-\frac{1}{\sigma}} \int_0^\infty K_t^{\chi(1-\frac{1}{\sigma})} dt.$$

As $\dot{K} = A_0 K^\pi = \hat{\beta} Y_0 K_0^{-\pi} K^\pi$, a change of variable yields

$$V(K_0, X_0) = \frac{u_0^{1-\frac{1}{\sigma}} K_0^{-\chi(1-\frac{1}{\sigma})}}{1-\frac{1}{\sigma}} \int_{K_0}^\infty \frac{K^{\chi(1-\frac{1}{\sigma})-\pi}}{\hat{\beta} Y_0 K_0^{-\pi}} dK.$$

This integral is finite if $\chi \left(1 - \frac{1}{\sigma}\right) - \pi < -1$, which, as we shall check, is the case if $\alpha > \hat{\beta}$ as we assume. Then

$$\begin{aligned}
V(K_0, X_0) &= \frac{u_0^{1-\frac{1}{\sigma}} K_0^{-\chi(1-\frac{1}{\sigma})}}{1 - \frac{1}{\sigma}} \frac{K_0^{-[\chi(\frac{1}{\sigma}-1)+\pi-1]}}{\hat{\beta} Y_0 K_0^{-\pi} [\chi(\frac{1}{\sigma}-1) + \pi - 1]} \\
&= \frac{1}{(1 - \frac{1}{\sigma}) \hat{\beta} [\chi(\frac{1}{\sigma}-1) + \pi - 1]} \frac{u_0^{1-\frac{1}{\sigma}} K_0}{Y_0} \\
&= \frac{1}{(1 - \frac{1}{\sigma}) \hat{\beta} [\chi(\frac{1}{\sigma}-1) + \pi - 1]} \frac{\left[(1 - \hat{\beta})^\omega Y_0^\omega X_0^{1-\omega}\right]^{1-\frac{1}{\sigma}} K_0}{Y_0} \\
&= \frac{(1 - \hat{\beta})^{\omega \frac{\sigma-1}{\sigma}}}{(1 - \frac{1}{\sigma}) \hat{\beta} [\chi(\frac{1}{\sigma}-1) + \pi - 1]} Y_0^{-\frac{\omega(1-\sigma)+\sigma}{\sigma}} X_0^{-\frac{(1-\omega)(1-\sigma)}{\sigma}} K_0 \\
&= B K_0^\xi X_0^\psi
\end{aligned}$$

with

$$\begin{aligned}
B &= - \frac{(1 - \hat{\beta})^{-\omega \frac{1-\sigma}{\sigma}} \left(\frac{\alpha - \hat{\beta}}{\hat{\phi}}\right)^{-\frac{\beta}{1-\beta} \frac{\omega(1-\sigma)+\sigma}{\sigma}}}{\frac{1-\sigma}{\sigma} \hat{\beta} [\chi \frac{1-\sigma}{\sigma} + \pi - 1]} < 0 \\
\xi &= 1 - \frac{\alpha - \beta}{1 - \beta} \frac{\sigma + (1 - \sigma)\omega}{\sigma}
\end{aligned} \tag{60}$$

$$\psi = - \frac{\beta}{1 - \beta} \frac{\sigma + (1 - \sigma)\omega}{\sigma} - \frac{(1 - \sigma)(1 - \omega)}{\sigma} \tag{61}$$

Coefficients ψ and ξ may be written in a more convenient way.

$$\begin{aligned}
\psi &= - \frac{\sigma + \omega(1 - \sigma)\omega}{\sigma(1 - \beta)} \beta \left[1 + \frac{(1 - \sigma)(1 - \omega)(1 - \beta)}{[\sigma + \omega(1 - \sigma)\omega] \beta} \right] \\
&= - \frac{\sigma + \omega(1 - \sigma)\omega}{\sigma(1 - \beta)} \beta \hat{\phi},
\end{aligned} \tag{62}$$

from (34).

$$\begin{aligned}
\xi &= \frac{(1-\beta)\sigma - (\alpha-\beta)(\omega(1-\sigma) + \sigma)}{(1-\beta)\sigma} \\
&= \frac{-(1-\beta)\omega(1-\sigma) + (1-\alpha)(\omega(1-\sigma) + \sigma)}{(1-\beta)\sigma} \\
&= -\frac{\omega(1-\sigma) + \sigma}{(1-\beta)\sigma} \left[\frac{(1-\beta)\omega(1-\sigma)}{\omega(1-\sigma) + \sigma} - 1 + \alpha \right] \\
&= -\frac{\omega(1-\sigma) + \sigma}{(1-\beta)\sigma} (\alpha - \hat{\beta}) = \frac{\alpha - \hat{\beta}}{\beta \hat{\phi}} \psi, \tag{63}
\end{aligned}$$

using (30).

This allows us to write the value function as in part (v) of the proposition.

Moreover, using (48), (45), (60), (61), (62), (63), and (30),

$$\begin{aligned}
&\chi \frac{1-\sigma}{\sigma} + \pi - 1 \\
&= \frac{\omega(1-\sigma) + \sigma}{\sigma} \pi - (1-\omega) \frac{\alpha - \hat{\beta}}{\hat{\beta} \hat{\phi}} \frac{1-\sigma}{\sigma} - 1 \\
&= \frac{\omega(1-\sigma) + \sigma}{\sigma} \left(\frac{\alpha - \beta}{\beta} - \frac{\alpha - \hat{\beta}}{\hat{\beta} \hat{\phi}} \right) \frac{\beta}{1-\beta} - (1-\omega) \frac{\alpha - \hat{\beta}}{\hat{\beta} \hat{\phi}} \frac{1-\sigma}{\sigma} - 1 \\
&= -1 + \frac{\omega(1-\sigma) + \sigma}{\sigma} \frac{\alpha - \beta}{1-\beta} - \left[\frac{\beta}{1-\beta} \frac{\omega(1-\sigma) + \sigma}{\sigma} + \frac{(1-\omega)(1-\sigma)}{\sigma} \right] \frac{\alpha - \hat{\beta}}{\hat{\beta} \hat{\phi}} \\
&= -\xi + \psi \frac{\alpha - \hat{\beta}}{\hat{\beta} \hat{\phi}} = -\psi \left[\frac{\alpha - \hat{\beta}}{\hat{\beta} \hat{\phi}} - \frac{\alpha - \hat{\beta}}{\hat{\beta} \hat{\phi}} \right] = -\psi \frac{\alpha - \hat{\beta}}{\hat{\beta} \hat{\phi}} \frac{\hat{\beta} - \beta}{\beta} \\
&= \frac{\sigma + \omega(1-\sigma)\omega}{\sigma(1-\beta)} \frac{\alpha - \hat{\beta}}{\hat{\beta}} (\hat{\beta} - \beta) = \frac{\alpha - \hat{\beta}}{\hat{\beta}}.
\end{aligned}$$

As $\psi < 0$, $\hat{\beta} > \beta$ and as we assume $\alpha > \hat{\beta}$, this expression is positive. This justifies the previous calculus of the value-function. As $\pi - 1 < 0$ and $1 - \sigma > 0$, this indeed proves that χ is positive. The consumption index u tends to infinity when capital, and time, tend to infinity which in turns ensures that instantaneous utility U tends to zero and that U^* is equal to zero.

We also obtain a simplified expression of the constant term of the value function:

$$B = -\frac{\sigma}{1-\sigma} \frac{(1-\hat{\beta})^{-\omega \frac{1-\sigma}{\sigma}}}{\alpha - \hat{\beta}} \left(\frac{\alpha - \hat{\beta}}{\hat{\phi}} \right)^{-\frac{\beta}{1-\beta} \frac{\sigma + (1-\sigma)\omega}{\sigma}}.$$

The impact of σ on u_0

Both $1 - \widehat{\beta}$ and $(\alpha - \widehat{\beta})/\widehat{\phi}$ are decreasing functions of $\widehat{\beta}$ and therefore decreasing functions of σ . Indeed

$$\frac{\alpha - \widehat{\beta}}{\widehat{\phi}} = \frac{\alpha - \widehat{\beta}}{1 + \frac{1-\widehat{\beta}}{\beta} \frac{1-\omega}{\omega}}$$

and

$$\frac{\partial}{\partial \widehat{\beta}} \left(\frac{\alpha - \widehat{\beta}}{\widehat{\phi}} \right) = - \left(1 + \frac{(1-\alpha)(1-\omega)}{\beta\omega} \right) \frac{1}{\widehat{\phi}^2} \leq 0$$

This implies (from equation (47)) that u_0 decreases with σ .